

## C.9 Plucked string instrument

The purpose of this lab is to experiment with models of a plucked string instrument, using it to deeply explore concepts of impulse response, frequency response, and spectrograms. The methods discussed in this lab were invented by Karplus and Strong, and first reported in

K. Karplus and A. Strong, “Digital Synthesis of Plucked-String and Drum Timbres,”  
*Computer Music Journal*, vol. 7, no. 2, pp. 43-55, Summer 1983.

The lab uses Simulink, like lab C.8. It assumes you have understood that lab and the techniques it uses in detail.

### C.9.1 Background

In the previous lab, you constructed in Simulink the feedback system shown in figure C.11, where  $S_2$  was an  $N$  sample delay. In this lab, you will enhance the model by replacing  $S_2$  with slightly more complicated filters. These filters will consist of the same  $N$  sample delay in cascade with two other filters, a **lowpass filter** and an **allpass filter**. The objective will be to get truly high-quality plucked string sounds.

#### Delays

Recall from example 8.6 that the  $N$  sample delay system has frequency response

$$H(\omega) = e^{-i\omega N}.$$

This same frequency response was obtained in example 9.10 by calculating the DTFT of the impulse response. Note that the magnitude response is particularly simple,

$$|H(\omega)| = 1.$$

Recall that this is an **allpass filter**.

The phase response is

$$\angle H(\omega) = -\omega N.$$

The phase response is a linear function of frequency,  $\omega$ , with slope  $-N$ . A filter with such a phase response is said to have **linear phase**. A delay is particularly simple form of a linear phase filter. Notice that the amount of delay is the negative of the derivative of the phase response,

$$\frac{d\angle H(\omega)}{d\omega} = -N.$$

This fact will be useful when we consider more complicated filters than this simple delay.

### Allpass Filters

We will need a slightly more complicated allpass filter than the  $N$  sample delay. Consider a filter given by the following difference equation,

$$\forall n \in \text{Integers}, \quad y(n) + ay(n-1) = ax(n) + x(n-1) \quad (\text{C.9})$$

for some constant  $0 < a \leq 1$ . This defines an LTI system, so if the input is  $x(n) = e^{j\omega n}$ , then the output is  $H(\omega)e^{j\omega n}$ , where  $H$  is the frequency response. We can determine the frequency response using this fact by plugging this input and output into (C.9),

$$H(\omega)e^{j\omega n} + aH(\omega)e^{j\omega(n-1)} = ae^{j\omega n} + e^{j\omega(n-1)}.$$

This can be rewritten as

$$H(\omega)e^{j\omega n}(1 + ae^{-j\omega}) = e^{j\omega n}(a + e^{-j\omega}).$$

Eliminating  $e^{j\omega n}$  on both sides we get

$$H(\omega)(1 + ae^{-j\omega}) = a + e^{-j\omega}.$$

Solving for  $H(\omega)$  we get

$$H(\omega) = \frac{a + e^{-j\omega}}{1 + ae^{-j\omega}}. \quad (\text{C.10})$$

We could immediately proceed to plotting the magnitude and phase response using Matlab, but instead, we will first manipulate this further to get some insight. Being slightly tricky, we will multiply top and bottom by  $e^{j\omega/2}$  to get

$$H(\omega) = \frac{ae^{j\omega/2} + e^{-j\omega/2}}{e^{j\omega/2} + ae^{-j\omega/2}}.$$

Now notice that the top and bottom are complex conjugates of one another. I.e., let

$$b(\omega) = ae^{j\omega/2} + e^{-j\omega/2} \quad (\text{C.11})$$

and note that

$$H(\omega) = \frac{b(\omega)}{b^*(\omega)}.$$

Since the numerator and denominator have the same magnitude,<sup>4</sup>

$$|H(\omega)| = 1.$$

The filter is allpass!

The phase response, however, is more complicated. Note that

$$\angle H(\omega) = \angle b(\omega) - \angle b^*(\omega).$$

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<sup>4</sup> For any two complex numbers  $z$  and  $w$ , note that  $|z/w| = |z|/|w|$  and  $\angle(z/w) = \angle(z) - \angle(w)$ .

But since for any complex number  $z$ ,  $\angle(z^*) = -\angle(z)$ ,

$$\angle H(\omega) = 2\angle b(\omega).$$

Thus, to find the phase response, we simply need to find  $\angle b(\omega)$ . Plugging Euler's relation into (C.11) we get

$$b(\omega) = (a + 1) \cos(\omega/2) + i(a - 1) \sin(\omega/2).$$

Since the angle of a complex number  $z$  is  $\tan^{-1}(\text{Im}\{z\}/\text{Re}\{z\})$ ,

$$\angle H(\omega) = 2 \tan^{-1} \left( \frac{(a - 1) \sin(\omega/2)}{(a + 1) \cos(\omega/2)} \right).$$

Since  $\tan(w) = \sin(w)/\cos(w)$ ,

$$\angle H(\omega) = 2 \tan^{-1} \left( \frac{a - 1}{a + 1} \tan(\omega/2) \right).$$

This form yields insight for small  $\omega$ . In particular, when  $\omega$  is small (compared to  $\pi$ ),

$$\tan(\omega/2) \approx \omega/2,$$

so

$$\angle H(\omega) \approx 2 \tan^{-1} \left( \frac{a - 1}{a + 1} \omega/2 \right).$$

Since  $0 < a \leq 1$ , the argument to the arctangent is small if  $\omega$  is small, so for low frequencies,

$$\angle H(\omega) \approx \frac{a - 1}{a + 1} \omega = -d\omega.$$

where  $d$  is defined by

$$d = -\frac{a - 1}{a + 1}. \quad (\text{C.12})$$

Thus, at low frequencies, this allpass filter has linear phase with slope  $-d$ . At low frequencies, therefore, it is an allpass with linear phase, which means that behave exactly like a delay! However, unlike the  $N$  sample delay, the amount of delay is  $d$ , which depending on  $a$  can be any real number between 0 and 1. Thus, this allpass filter gives us a way to get fractional sample delays in a discrete time system, at least at low frequencies.

## C.9.2 In-lab section

1. The lowpass filter we will use is a simple, length two moving average. If the input is  $x$  and the output is  $y$ , then the filter is given by the difference equation,

$$\forall n \in \text{Integers}, \quad y(n) = 0.5(x(n) + x(n - 1)). \quad (\text{C.13})$$

Find an expression for the frequency response of the lowpass filter given by (C.13). Use Matlab to plot the magnitude and phase response over the frequency range 0 to  $\pi$  radians/sample. Is this a linear phase filter? If so, what is its delay?

2. In part 4 of the previous lab, you initialized a comb filter with random noise and produced a sound that reasonably well approximates a plucked string instrument, such as a guitar. We can improve the sound.

Real instrument sounds have more dynamics in their frequency structure. That is, the spectrum of the sound within the first few milliseconds of plucking the string is different from the spectrum a second or so later. Physically, this is because the high frequency vibrations of the string die out more rapidly than the low frequency vibrations.

We can approximate this effect by modifying the comb filter by inserting the lowpass filter given by (C.13) into the feedback loop. This can be accomplished by realizing the following difference equation:

$$\forall n \in \text{Integers}, \quad y(n) = x(n) + 0.5\alpha(y(n - N) + y(n - N - 1)).$$

Modify your Simulink model you constructed in part 4 of the previous lab so that it uses a lowpass filter in the feedback loop, implementing this difference equation. Listen to the resulting sound, and compare it against the sound from the previous lab. Use  $\alpha = 0.99$  and  $N = 40$ , as before. Can you hear the improvement?

3. In the last lab, you found that the tone of the sound generated by the comb filter had frequency  $8000/N$ , where  $N$  was the delay in the feedback loop, and 8000 was the sampling frequency. You used  $N = 40$  to get a fundamental frequency of 200 Hz. Now, you have added an additional lowpass filter, which introduces additional delay in the feedback loop. You have determined that additional delay in part 1 above. What is the fundamental frequency now?

The comb filter delay can only delay by an integer number of samples. The lowpass filter introduces a fixed delay. Consequently, there are only certain fundamental frequencies that are possible. In particular, assuming the sample rate is 8 kHz, is it possible to achieve a fundamental frequency of 440 Hz? This would be essential to have a reasonable guitar model, since we would certainly want to be able to play the note A-440 on the guitar. Determine the closest achievable frequency to 440 Hz. Is it close enough? In the independent section of this lab, you will show how to achieve a fundamental frequency very close to 440 Hz.

### C.9.3 Independent section

1. Show analytically that the lowpass filter given by (C.13) has linear phase over the range of frequencies 0 to  $\pi$  radians/sample, and determine the slope. Verify that this agrees with the plot you constructed in the In-Lab section.
2. In part 2 of the in-lab section, you combined an  $N$ -sample delay with a lowpass filter in the feedback path of a comb filter. Calculate the frequency response of this version of the comb filter, and plot its magnitude using Matlab over the frequency range 0 to  $\pi$ . Compare it to the frequency response you calculated for the original comb filter in the previous lab. Find the fundamental frequency of the musical note from this plot and compare it to the answer that you gave in part 3 of the in-lab portion. **Hint:** The spectral peaks are very sharp, so you will need to calculate the magnitude frequency at many points in the range 0 to  $\pi$  to be sure to hit the peaks. We recommend calculating at least 2000 points.

3. The reason that the comb filter with a lowpass filter in the feedback loop yields a much better plucked string sound than the comb filter by itself is that it more accurately models the physical phenomenon that higher frequency vibrations on the string die out faster than lower frequency vibrations. Plot the spectrogram using `specgram` of the generated sound to demonstrate this phenomenon, and explain how your spectrogram demonstrates it.
4. Verify that the frequency response (C.10) of the allpass filter has constant magnitude and linear phase for low frequencies by plotting it using Matlab. Plot it for the following values of delay:  $d = 0.1, 0.4, 0.7, \text{ and } 1.0$ . Plot it over the range of frequencies 0 to  $\pi$  radians/sample. Discuss how your plots support the conclusions about this filter. **Hint:** Use (C.12) to find  $a$  given  $d$ .
5. You determined in part 3 of the in-lab section that you could not get very close to A-440 with a comb filter with a lowpass filter in the feedback loop. The allpass filter given by (C.10), however, can be used to achieve delays that are a fraction of a sample period. Implement the allpass filter, modifying your Karplus-Strong plucked string model by putting it in the feedback loop. Set the parameters of the allpass filter and  $N$  to get an A-440. Show your Simulink diagram, and give the parameters of all the blocks.

## Instructor Verification Sheet for C.9

Name: \_\_\_\_\_ Date: \_\_\_\_\_

1. Magnitude and phase of lowpass filter. Linear phase? Delay?

**Instructor verification:** \_\_\_\_\_

2. Improved plucked string sound.

**Instructor verification:** \_\_\_\_\_

3. Fundamental frequencies that are possible.

**Instructor verification:** \_\_\_\_\_