

C.5 Difference equations

The purpose of this lab is to construct difference equation models of systems and to study their properties. In particular, we experimentally examine **stability** by constructing stable, unstable, and marginally stable systems. We will also introduce elementary complexity measures. The principal new Matlab skill required to develop these concepts is matrix operations.

C.5.1 In-lab section

1. Matlab is particularly good at matrix arithmetic. In this problem, we explore matrix multiplication (see sidebar on page 151).

- (a) Consider the 2×2 matrix

$$M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Without using Matlab, give M^n , for $n = 0, 1, 2, 3$. Recall that by mathematical convention, for any square matrix M , $M^0 = I$, the identity matrix, so in this case,

$$M^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Guess the general form of the matrix M^n . That is, give an expression for each of the elements of the matrix M^n .

- (b) Use Matlab to compute M^{25} . Was your guess correct? Calculate a few more values using Matlab until your guess is correct.
 - (c) If your guess was correct, try to show it using induction. That is, first show that your guess for M^n is correct for some fixed n , like for example $n = 0$. Then assume your guess for M^n is correct is for some fixed n , and show that it is correct for M^{n+1} .
2. A vector is a matrix where either the number of rows is one (in the case of a **row vector**) or the number of columns is one (in the case of a **column vector**). Let

$$b = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

be a column vector. We can equally well write this $b = [2, 3]^T$, where the superscript T indicates that the row vector $[2, 3]$ is transposed to make a column vector.

- (a) Create a column vector in Matlab equal to b above. Multiply it by M , given in the previous problem. Try forming both bM and Mb . Why does only one of these two work?
- (b) Create a row vector by transposing b . (Try help transpose or look up “transpose” in the help desk.) Multiply this transpose by M . Try both $b^T M$ and $M b^T$. Why does only one of them work?

3. Consider a 2-dimensional difference equation system given by

$$A = \sigma \begin{bmatrix} \cos(\omega) & -\sin(\omega) \\ \sin(\omega) & \cos(\omega) \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad c = \sigma \begin{bmatrix} -\cos(\omega) \\ \sin(\omega) \end{bmatrix}, \quad d = 0,$$

where $\omega, \sigma \in \text{Reals}$. Note that this is similar to the systems studied in exercises 10 and 14 of chapter 5, with the differences being the multiplying constant σ and the c vector. Let $\omega = \pi/8$ and plot the first 100 samples of the zero-state impulse response for the following values of σ .

- $\sigma = 1$.
- $\sigma = 0.95$.
- $\sigma = 1.05$.
- For which values of σ is the result periodic? What is the period? The system producing the periodic output is called an **oscillator**.
- You have constructed three distinct difference equation systems. One of these is a **stable system**, one is an **unstable system**, and one is a **marginally stable**. Which is which? You can infer the answer from the ordinary English-language meaning of the word “stable.” What will happen if the unstable system is allowed to continue to run beyond the 100 samples you calculated?

C.5.2 Independent section

1. In lab C.1 you constructed a sound waveform $f: \text{Reals} \rightarrow \text{Reals}$ given by

$$\forall t \in [0, 1], \quad f(t) = \exp(-5t) \sin(2\pi \times 440t).$$

You wrote a Matlab script that calculated samples of this waveform at a sample rate of 8 kHz. In this lab, we will construct the same waveform in a very different way, using difference equations.

Construct a difference equation system with impulse response given by

$$\forall n \in \text{Naturals}_0, \quad h(n) = \exp(-5n/8000) \sin(2\pi \times 440n/8000).$$

Give the matrix A , the vectors b , and c , and the scalar d of (5.35) and (5.36). Also give a Matlab program that produces the first 8000 samples of this impulse response and plays it as a sound. **Hint:** You will need to understand what you did in problem 3 of the in-lab section, and you may find it useful to use the results of exercise 14 in chapter 5.

2. For the system with the impulse response constructed in part 1, change the input so it consists of an impulse every 1/5 of a second. I.e., at an 8kHz sample rate,

$$x(n) = \begin{cases} 1 & \text{if } n \text{ is a multiple of } 1600 \\ 0 & \text{otherwise} \end{cases}$$

Write a Matlab script that plays two seconds of sound with this input. **NOTE:** This is a simplified model of a guitar string being repeatedly plucked. The model is justifiable on physical grounds, although it is a fairly drastic simplification.

3. Compare the complexity of the state machine model and the one you constructed in lab C.1. In particular, assuming in each case that you generate one second of sound at an 8kHz sample rate, count the number of scalar multiplications and additions that must be done to construct the sound vector. In the realization in lab C.1, you used the built-in Matlab functions `exp` and `sin`. These functions are surprisingly expensive to compute, so count each evaluation of `exp` or `sin` on a scalar argument as 20 multiplications and 15 additions (they are actually typically more expensive even than this). You should find that the state machine realization is far less expensive by this measure. Do not count the cost of the Matlab `sound` function, which we can't easily determine.

Instructor Verification Sheet for C.5

Name: _____ Date: _____

1. Matrix multiplication in Matlab, and induction demonstration.

Instructor verification: _____

2. Matrix-vector multiplication.

Instructor verification: _____

3. Sinusoids with exponential envelopes; stability.

Instructor verification: _____